Exam I: MTH 213, Spring 2018

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Score = $\frac{64}{64}$

QUESTION 1. (i) **(5 points)** Prove that $\sqrt{55}$ is irrational. (Hint: You must use this technique: Deny. Then $\sqrt{55} = a/b$ for some positive ODD integers a, b s.t. $gcd(a, b) = 1$, now start cooking as explained in the class)

Deny; say $\sqrt{55}$ is rational.

$\sqrt{55} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b) = 1$.

a and b are odd integers, let $a = 2m+1$, $b = 2n+1$., $m, n \in \mathbb{Z}$.

$\sqrt{55} = \frac{2m+1}{2n+1}$

$55 = \frac{4m^2+4m+1}{4n^2+4n+1}$

$55(4n^2+4n+1) = 4m^2+4m+1$

$55(4n^2+4n) + 54 = 4m^2+4m$

$\underbrace{55n^2 + 55n + \frac{54}{4}}_{\notin \mathbb{Z}} = \underbrace{m^2+m}_{\text{integer.}}$ Contradiction. Hence $\sqrt{55}$ is irrational.

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(ii) **(3 points)** Prove that $\sqrt{5} + \sqrt{11}$ is irrational (Hint: You may use the result from (i))

Deny. Say $\sqrt{5} + \sqrt{11}$ are rational.

$\sqrt{5} + \sqrt{11} = \frac{a_0}{b_0}$ $a_0, b_0 \in \mathbb{Z}$, $b_0 \neq 0$, $\gcd(a_0, b_0) = 1$.

$(\sqrt{5} + \sqrt{11})^2 = \left(\frac{a_0}{b_0}\right)^2$

$5 + 2\sqrt{55} + 11 = \frac{a_0^2}{b_0^2}$

$\sqrt{55} = \frac{a_0^2}{2b_0^2} - \frac{16}{2}$. LHS is irrational as shown in (i). RHS is rational. Contradiction. Hence, $\sqrt{5} + \sqrt{11}$ is irrational.

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QUESTION 2. (i) **(6 points)** For every $n \geq 1$, use math induction to prove that $18 \mid (5^{6n} - 1)$.

$\boxed{1}$ Prove for $n = 1$.

$5^6 - 1 = 15624$, $18 \mid 15624$.

$\boxed{2}$ Assume : $18 \mid 5^{6n} - 1$ for some $n \geq 1$

$\boxed{3}$ Prove for $n+1$.

$5^{6n+6} - 1$

$= 5^{6n} \cdot 5^6 - 1$

$= 5^{6n} \cdot 5^6 - 5^6 + 5^6 - 1$

$= \underbrace{5^6(5^{6n} - 1)}_{\text{divisible by 18, as shown in } \boxed{2}} + \underbrace{5^6 - 1}_{\text{divisible by 18, as shown in } \boxed{1}}$

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Hence $18 \mid 5^6(5^{6n} - 1) + 5^6 - 1 \Rightarrow 18 \mid 5^{6n} - 1$.

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(ii) **(3 points)** Use direct proof to show that $18 \mid (5^{6n} - 1)$, for every $n \geq 1$.

$18 = 2 \times 3^2$, $\phi(18) = 1 \times 2 \times 3 = 6$. $\gcd(5, 18) = 1$.

By Euler Fermat result, $5^6 \equiv 1 \pmod{18}$

Multiplying 5^6 n times: $(5^6)^n \equiv 1^n \pmod{18}$

$5^{6n} \equiv 1 \pmod{18}$.

Hence $18 \mid 5^{6n} - 1$.

QUESTION 3. (i) **(5 points)** Use math induction to prove that $\sum_{i=0}^{n} \frac{1}{(i+4)(i+5)} = \frac{n+1}{4n+20}$ for every $n \geq 1$

$\boxed{1}$ Prove for $n = 1$.

$\sum_{i=0}^{1} \frac{1}{(i+4)(i+5)} = \frac{1}{20} + \frac{1}{30} = \frac{1}{12}$. check $\frac{n+1}{4n+20} = \frac{2}{24} = \frac{1}{12}$

$\boxed{2}$ Assume: $\sum_{i=0}^{n} \frac{1}{(i+4)(i+5)} = \frac{n+1}{4n+20}$ for some $n \geq 1$.

$\boxed{3}$ Prove for $n+1$.

better if you write we show $\sum_{i=0}^{n+1} \frac{1}{(i+4)(i+5)} = \frac{n+2}{4n+24}$

$$\sum_{i=0}^{n+1} \frac{1}{(i+4)(i+5)} = \sum_{i=0}^{n} \frac{1}{(i+4)(i+5)} + \frac{1}{(n+5)(n+6)}$$

$$= \frac{n+1}{4n+20} + \frac{1}{(n+5)(n+6)}$$

$$= \frac{n+1}{4(n+5)} + \frac{1}{(n+5)(n+6)}$$

$$= \frac{(n+1)(n+6) + 4}{4(n+5)(n+6)} = \frac{n^2 + 7n + 10}{4(n+5)(n+6)} = \frac{(n+2)\cancel{(n+5)}}{4\cancel{(n+5)}(n+6)} = \frac{n+2}{4n+24}$$

$\sum_{i=0}^{n+1} \frac{1}{(i+4)(i+5)} = \frac{(n+1)+1}{4(n+1)+20}$, hence $\sum_{i=0}^{n} \frac{1}{(i+4)(i+5)} = \frac{n+1}{4n+20}$, $\forall n \geq 1$.

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(ii) **(3 points)** Use direct proof to show that $\sum_{i=0}^{n} \frac{1}{(i+4)(i+5)} = \frac{n+1}{4n+20}$ (Hint: First note that $\frac{1}{(i+4)(i+5)} = \frac{1}{i+4} - \frac{1}{i+5}$. For each $0 \leq i \leq n$, let $a_i = \frac{1}{i+4} - \frac{1}{i+5}$. Now calculate $a_0 + a_1 + \cdots + a_n$ and stare, you should observe something.!)

$\sum_{i=0}^{n} \frac{1}{(i+4)(i+5)} = \sum_{i=0}^{n} \left[\frac{1}{(i+4)} - \frac{1}{(i+5)}\right] = a_0 + a_1 + \dots a_n$ where $a_i = \frac{1}{(i+4)} - \frac{1}{(i+5)}$

$\sum_{i=0}^{n} \frac{1}{(i+4)(i+5)} = \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} \dots + \frac{1}{n+3} - \frac{1}{n+4} + \frac{1}{n+4} - \frac{1}{n+5}$

telescopic?

Note all terms except first and last cancel out

$\sum_{i=0}^{n} \frac{1}{(i+4)(i+5)} = \frac{1}{4} - \frac{1}{(n+5)} = \frac{n+5-4}{4(n+5)} = \frac{n+1}{4n+20}$. 3/3

QUESTION 4. **(3 points)** Find $(265)_7 \times (56)_7$

$$\begin{array}{r} 2\ 6\ 5 \\ \times\quad 5\ 6 \\ \hline 2\ 3\ 5\ 2 \\ +\ 2\ 0\ 5\ 4\ \\ \hline (2\ 3\ 2\ 2\ 2)_7 \end{array}$$

(mod 7) (div 7)

Ans: $(23222)_7$

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QUESTION 5.

(3 points) Find $(1055)_9 - (338)_9$

$$\begin{array}{r} 1\ 0\ 5\ 5 \\ -\quad 3\ 3\ 8 \\ \hline 6\ 1\ 6 \end{array}$$

(mod 9) (div 9)

Ans: $(616)_9$

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QUESTION 6. (4 points) JUST WRITE T OR F

(i) $\exists! x \in Z$ such that $\forall y \in R$, $x + y = y$ T

(ii) $\forall x \in Z_6^*$, $\exists y \in Z_6^*$ such that $xy = 1$ over planet Z_6 (note $Z_6^* = \{1, 2, 3, 4, 5\}$) F

(iii) $\forall x \in Z_6^*$, $\exists! y \in Z_6^*$ such that $xy = 1$ over planet Z_6 (note $Z_6^* = \{1, 2, 3, 4, 5\}$) F

(iv) $\exists! x \in Q^*$ such that $2x^2 + 3x + 1 = 0$ F not unique

A+

QUESTION 7. (8 points) Let $d = gcd(98, 119)$. Find d over PLANET N. Then find two integers in PLANET Z, say m, n, such that $d = 98n + 119m$.(Show the work)

gcd (98, 119)

$$\begin{array}{r|l} & 1 \\ 98 & 119 \\ & -98 \\ & 21 \end{array} \rightarrow \begin{array}{r|l} & 4 \\ 21 & 98 \\ & -84 \\ & 14 \end{array} \rightarrow \begin{array}{r|l} & 1 \\ 14 & 21 \\ & -14 \\ & 7 \end{array} \rightarrow \begin{array}{r|l} & 2 \\ 7 & 14 \\ & -14 \\ & 0 \end{array}$$

$gcd(98, 119) = 7 \in N$

$$\begin{aligned} 7 &= 21 - 14 \\ &= 21 - (98 - 21(4)) \\ &= 21 - 98 + 21(4) \\ &= 5(21) - 98 \\ &= 5(119 - 98) - 98 \\ &= 5(119) - 5(98) - 98 \\ 7 &= 5(119) - 6(98) \end{aligned}$$

$n = -6 \quad m = 5 \quad , \ n, m \in Z.$

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QUESTION 8. **(5 points)** Find two numbers say n, m such that there are 2018 consecutive non-prime integers, say $a_1, a_2, ..., a_{2018}$, where $n < a_i < m$, for each $1 \le i \le 2018$. Then find a_1, a_2, a_{2018}. (Hint: write your solution in terms of Factoria, i.e., you may say $n = 23! + 7$ and so on)

For any number x; $(2x)! + x, (2x)! + x+1, (2x)! + x+2, \ldots, (2x)! + 2x$ are definitely not prime. The size of this set is x. Since 2018 consecutive non-prime integers are required let $n = (4036)! + 2017$ and $m = (4036)! + 4037$

Then $a_1 = 4036! + 2018$

$a_2 = 4036! + 2019$

$a_{2018} = 4036! + 4036$

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QUESTION 9. (i) **(5 points)** Solve $6x = 3$ over planet Z_9.

$6x = 3$ in Z_9 gcd $(6,9) = 3$ ~~∤~~ $3 | 3$.

$\therefore$ 3 sol°.

$x = 2$, $x = 5$, $x = 8$

(ii) **(3 points)** Solve over planet Z, $6x \equiv 3 \pmod 9$

$x = 2 + 9k_1$, $x = 5 + 9k_2$, $x = 8 + 9k_3$

$k_1, k_2, k_3 \in Z$.

QUESTION 10. **(8 points)** Let X be the number of females in some sport-activity at the AUS. Given $X \equiv 2 \pmod 4$, $X \equiv 5 \pmod 9$, and $X \equiv 10 \pmod{11}$. If $0 < X < 396$, then find X. (Show the work)

$x \equiv \underset{r_1}{2} \pmod{\underset{m_1}{4}}$ $x \equiv \underset{r_2}{5} \pmod{\underset{m_2}{9}}$ $x \equiv \underset{r_3}{10} \pmod{\underset{m_3}{11}}$

$(m_2 m_3)^{-1} \bmod m_1$ $(m_1 m_3)^{-1} \bmod m_2$ $(m_1 m_2)^{-1} \bmod m_3$

$99x = 1$ in Z_4 $44x \equiv 1 \pmod 9$ $36x \equiv 1 \pmod{11}$

$3x = 1$ in Z_4 $8x \equiv 1 \pmod 9$ $3x \equiv 1 \pmod{11}$

$\boxed{x = 3 = d_1}$ $\boxed{x = 8 = d_2}$ $\boxed{x = 4 = d_3}$

$X = 99(3)(2) + 44(8)(5) + 36(4)(10)$

$= 3794 \bmod 396$

$X = 230$, $0 < X < 396$.

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